The Power of Whispers:
A Theory of Rumor, Communication and Revolution

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Technical Appendix
(Not intended for publication)

A. Lemmas

Lemma 1. There exists a unique neutral rumor z' such that $\theta_m^*(z') = \theta_{ps}^*(z') = \theta_{ms}^*$ and $\theta_{ps}^*(z) > \theta_{ms}^*$ for all $z < z'$ and $\theta_{ps}^*(z) < \theta_{ms}^*$ for all $z > z'$.

Proof. Since $(\theta_{ms}^*, x_{ms}^*)$ satisfies the indifference condition of the "pure noise model," we have

$$\Phi\left(\frac{\theta_{ms}^* - x_{ms}^*}{\sigma_x}\right) = c.$$

By definition, when $z = z'$, $(\theta_{ms}^*, x_{ms}^*)$ also satisfies the indifference condition of the "public signal model," we have

$$\Phi\left(\frac{\theta_{ms}^* - (\beta x_{ms}^* + (1 - \beta)z')}{\sqrt{\beta}\sigma_x}\right) = c.$$

The posterior belief $P(\cdot|z', x)$ in the "mute model" is just the weighted average of the left-hand-side of the two equations above. Hence $P(\theta_{ms}^*|z', x_{ms}^*) = c$. Furthermore, $(\theta_{ms}^*, x_{ms}^*)$ satisfies the critical mass condition of the "mute model." Therefore, it is an equilibrium for the "mute model" when $z = z'$. Since $\theta_{ps}^*(z)$ is strictly decreasing, the second part of this lemma is implied. ∎

Lemma 2. At $z = z'$ and $\hat{\theta} = \theta'$, the cutoff types who are indifferent between attacking and not attacking satisfy:

$$\frac{\partial \hat{x}_I}{\partial z} < \frac{\partial \hat{x}_m}{\partial z} < \frac{\partial \hat{x}_U}{\partial z} < 0,$$
$$\frac{\partial \hat{x}_I}{\partial \hat{\theta}} > \frac{\partial \hat{x}_m}{\partial \hat{\theta}} > \frac{\partial \hat{x}_U}{\partial \hat{\theta}} > 0.$$

Proof. We proceed in a number of steps.

Claim 1. $\Pr[y_i = 1|z, x_i]$ is increasing in x_i for $x_i < z$ and decreasing in x_i for $x_i > z$.

The derivative of $\Pr[y_i = 1|z, x_i]$ with respect to x_i is:

$$- w \frac{\beta}{\sqrt{1 + \beta}\sigma_x} \left[\phi\left(\frac{\overline{x}(z) - X_i}{\sqrt{1+\beta}\sigma_x}\right) - \phi\left(\frac{\underline{x}(z) - X_i}{\sqrt{1+\beta}\sigma_x}\right) \right]$$
$$- (1-w) \frac{1}{\sqrt{2}\sigma_x} \left[\phi\left(\frac{\overline{x}(z) - x_i}{\sqrt{2}\sigma_x}\right) - \phi\left(\frac{\underline{x}(z) - x_i}{\sqrt{2}\sigma_x}\right) \right] + \frac{\partial w}{\partial x_i} F(\beta, z, x_i),$$

where

$$F(\beta, z, x_i) = \Phi\left(\frac{\overline{x}(z) - X_i}{\sqrt{1+\beta}\sigma_x}\right) - \Phi\left(\frac{\underline{x}(z) - X_i}{\sqrt{1+\beta}\sigma_x}\right) - \Phi\left(\frac{\overline{x}(z) - x_i}{\sqrt{2}\sigma_x}\right) + \Phi\left(\frac{\underline{x}(z) - x_i}{\sqrt{2}\sigma_x}\right).$$

It is straightforward to show that $F(\beta, z, x_i)$ is decreasing in β, with $F(1, z, x_i) = 0$. Thus, $F(\beta, z, x_i) > 0$ for $\beta < 1$. Since $\overline{x}(z) + \underline{x}(z) = 2z$, the first two terms are positive if $x_i < z$. Since $\partial w / \partial x_i > 0$ for $x_i < z$, the third term is positive as well. Therefore, the derivative is positive. If $x_i > z$, then the opposite is true.

Claim 2. When $c = 0.5$,

$$\frac{\partial \hat{x}_I(\theta', z')}{\partial \hat{\theta}} = \frac{-p(\theta'|z', x')}{\int_{-\infty}^{\theta'} \frac{J(t,z')}{J(\theta',z')} \frac{\partial p(t|z',x')}{\partial x} dt}, \qquad \frac{\partial \hat{x}_U(\theta', z')}{\partial \hat{\theta}} = \frac{-p(\theta'|z', x')}{\int_{-\infty}^{\theta'} \frac{1-J(t,z')}{1-J(\theta',z')} \frac{\partial p(t|z',x')}{\partial x} dt}.$$

When $c = 0.5$, the value of x' that satisfies $P(\theta'|z', x', 1) = c$ is $x' = z'$. By Claim 1, $\partial \Pr[y_i = 1|z', x'] / \partial x_i = 0$ at this point. Since

$$P(\theta'|z', x', 1) = \frac{\int_{-\infty}^{\theta'} J(t, z') p(t|z', x') \, dt}{\Pr[y_i = 1|z', x']},$$

the claim follows by the implicit function theorem. The expression for $\partial x_U / \partial \hat{\theta}$ is derived in a similar fashion.

Claim 3. When $c = 0.5$,

$$\int_{-\infty}^{\theta'} \frac{J(t,z')}{J(\theta',z')} \frac{\partial p(t|z',x')}{\partial x} dt > \int_{-\infty}^{\theta'} \frac{\partial p(t|z',x')}{\partial x} dt > \int_{-\infty}^{\theta'} \frac{1-J(t,z')}{1-J(\theta',z')} \frac{\partial p(t|z',x')}{\partial x} dt.$$

When $c = 0.5$, $\theta' = z' = x'$. Therefore, $J(t, z') < J(\theta', z')$ for all $t < \theta'$. Moreover,

$$\frac{\partial p(t|z', x')}{\partial x} = -w \frac{1}{\sigma_x^2} \phi_I' - (1-w) \frac{1}{\sigma_x^2} \phi_U' + \frac{\partial w}{\partial x} \left(\frac{1}{\sqrt{\beta}\sigma_x} \phi_I - \frac{1}{\sigma_x} \phi_U \right),$$

where the subscript I indicates that the corresponding function is evaluated at $(t - X')/(\sqrt{\beta}\sigma_x)$. Since $t < \theta' = X'$, ϕ_I' is positive. Likewise, the subscript U indicates

that the corresponding function is evaluated at $(t - x')/\sigma_x$. Since $t < \theta' = x'$, ϕ'_U is also positive. Finally, since $\partial w/\partial x = 0$ at $x' = z'$, we have $\partial p(t|z', x')/\partial x < 0$. Thus, the first inequality of the claim follows. The second inequality can be established in a similar way.

Since x' satisfies the indifference condition $P(\theta'|z', x') = c$ of the "mute model," by the implicit function theorem we obtain:

$$\frac{\partial \hat{x}_m(\theta', z')}{\partial \hat{\theta}} = \frac{-p(\theta'|z', x')}{\int_{-\infty}^{\theta'} \frac{\partial p(\theta|z', x')}{\partial x}\, d\theta} = \frac{-p(\theta'|z', x')}{P_x}.$$

The ranking of the partial derivatives in the lemma then follows by Claims 2 and 3. Finally, since $P(\hat{\theta}|z, x, 0)$ increases in $\hat{\theta}$ and decreases in x, we have $\partial \hat{x}_U/\partial \hat{\theta} > 0$. This proves the second inequality in the lemma. The first inequality then follows immediately from Lemma 6, which shows that $\partial \hat{x}/\partial z = 1 - \partial \hat{x}/\partial \hat{\theta}$. ∎

Lemma 3. $\lim_{z \to -\infty} x_I^*(z) = +\infty$ and $\lim_{z \to \infty} x_I^*(z) = -\infty$.

Proof. We only prove the first part of this lemma; the proof of the second part is similar.

Claim 1. (a) $\lim_{z \to -\infty} \underline{x}(z)/z = 1 + \sigma_I/\sigma_U$; and (b) $\lim_{z \to -\infty} \overline{x}(z)/z = 1 - \sigma_I/\sigma_U$.

Recall that $\overline{x}(z)$ is the larger solution to $w(z, x) = \delta$. Solving this equation gives

$$\overline{x}(z) = z + \sqrt{\frac{\sigma_I^2}{\sigma_U^2}(z - s)^2 - 2\sigma_I^2 \log \frac{\sigma_I \delta(1 - \alpha)}{\sigma_U \alpha(1 - \delta)}} \equiv z + \kappa(z).$$

Since $\lim_{z \to \infty} \kappa(z)/z = \sigma_I/\sigma_U$, this establishes part (a). Part (b) also follows because $\underline{x}(z) = z - \kappa(z)$.

Claim 2. For any finite $\hat{\theta}$ and any $x_I^* \neq \infty$, $\lim_{z \to -\infty} P(\hat{\theta}|z, x_I^*, y_i = 1, z \sim I) = 1$.

Let $\lim_{z \to -\infty} x_I^*(z)/z = \gamma \geq 0$. Consider the complementary probability,

$$\Pr[\theta > \hat{\theta}|z, x_I^*, y_i = 1, z \sim I] = \frac{\int_{\hat{\theta}}^{\infty} J(t, z) \frac{1}{\sqrt{\beta}\sigma_x} \phi\left(\frac{t - z - \beta(x_I^* - z)}{\sqrt{\beta}\sigma_x}\right) dt}{\Phi\left(\frac{\overline{x}(z) - z - \beta(x_I^* - z)}{\sqrt{1 + \beta}\sigma_x}\right) - \Phi\left(\frac{\underline{x}(z) - z - \beta(x_I^* - z)}{\sqrt{1 + \beta}\sigma_x}\right)}.$$

Since $J(t, z)$ is decreasing in t for $t > z$, we have

$$\lim_{z \to -\infty} \Pr[\theta > \hat{\theta}|z, x_I^*, y_i = 1, z \sim I] \leq \lim_{z \to -\infty} \frac{J(\hat{\theta}, z)\left(1 - \Phi\left(\frac{\hat{\theta} - z - \beta(x_I^* - z)}{\sqrt{\beta}\sigma_x}\right)\right)}{\Phi\left(\frac{\overline{x}(z) - z - \beta(x_I^* - z)}{\sqrt{1 + \beta}\sigma_x}\right) - \Phi\left(\frac{\underline{x}(z) - z - \beta(x_I^* - z)}{\sqrt{1 + \beta}\sigma_x}\right)}.$$

Note that $\lim_{z\to-\infty}(\overline{x}(z) - z - \beta(x - z))/z = \beta(1 - \gamma) - \sigma_I/\sigma_U$. There are two cases to consider. If $\beta(1 - \gamma) - \sigma_I/\sigma_U \le 0$, then the denominator of the term above does not vanish as z goes to minus infinity, while the numerator goes to zero. So the limit of the ratio is 0. If $\beta(1 - \gamma) - \sigma_I/\sigma_U > 0$ then both denominator and numerator vanishes. However, $J(\hat{\theta}, z)$ goes to 0 at the rate at which $(\overline{x}(z) - \hat{\theta})/\sigma_x$ goes to minus infinity, which is equal to $(1 - \sigma_I/\sigma_U)/\sigma_x$. The denominator goes to 0 at the rate at which $(\overline{x}(z) - z - \beta(x - z))/(\sqrt{1 + \beta}\sigma_x)$ goes to minus infinity, which is

$$\frac{\beta(1 - \gamma) - \sigma_I/\sigma_U}{\sqrt{1 + \beta}\sigma_x} < \frac{1 - \sigma_I/\sigma_U}{\sigma_x}.$$

Hence, in both cases,

$$\lim_{z\to-\infty} \frac{J(\hat{\theta}, z)}{\Phi\left(\frac{\overline{x}(z) - z - \beta(x_I^* - z)}{\sqrt{1 + \beta}\sigma_x}\right) - \Phi\left(\frac{\underline{x}(z) - z - \beta(x_I^* - z)}{\sqrt{1 + \beta}\sigma_x}\right)} = 0.$$

This implies that $\lim_{z\to-\infty} P(\hat{\theta}|z, x_I^*, y_i = 1, z \sim I) = 1$.

Claim 3. For any finite $\hat{\theta}$ and any $x_I^* \ne \infty$, $\lim_{z\to-\infty} P(\hat{\theta}|x_I^*, y_i = 1, z \sim U) = 1$.

Let $\lim_{z\to-\infty} x_I^*(z)/z = \gamma \ge 0$. Consider limit of the complementary probability,

$$\lim_{z\to-\infty} \Pr[\theta > \hat{\theta}|x_I^*, y_i = 1, z \sim U] = \lim_{z\to\infty} \frac{\int_{\hat{\theta}}^{\infty} J(t, z)\frac{1}{\sigma_x}\phi\left(\frac{t - x_I^*}{\sigma_x}\right) dt}{\Phi\left(\frac{\overline{x}(z) - x_I^*}{\sqrt{2}\sigma_x}\right) - \Phi\left(\frac{\underline{x}(z) - x_I^*}{\sqrt{2}\sigma_x}\right)}$$

$$\le \lim_{z\to-\infty} \frac{J(\hat{\theta}, z)\left(1 - \Phi\left(\frac{\hat{\theta} - x_I^*}{\sigma_x}\right)\right)}{\Phi\left(\frac{\overline{x}(z) - x_I^*}{\sqrt{2}\sigma_x}\right) - \Phi\left(\frac{\underline{x}(z) - x_I^*}{\sqrt{2}\sigma_x}\right)}.$$

The term $J(\hat{\theta}, z)$ goes to 0 at the rate $(1 - \sigma_I/\sigma_U)/\sigma_x$. If $1 - \sigma_I/\sigma_U > \gamma$, the denominator goes to 0 at the rate

$$\frac{1 - \gamma - \sigma_I/\sigma_U}{\sqrt{2}\sigma_x} < \frac{1 - \sigma_I/\sigma_U}{\sigma_x}.$$

Therefore the ratio goes to 0 as z goes to minus infinity. If $1 - \sigma_I/\sigma_U \le \gamma$, the denominator does not vanish. So the ratio again goes to zero.

To prove the lemma, note that $P(\hat{\theta}|z, x_I^*, 1)$ is just a weighted average of $P(\hat{\theta}|z, x_I^*, y_i = 1, z \sim I)$ and $P(\hat{\theta}|z, x_I^*, y_i = 1, z \sim U)$. By Claims 2 and 3, we must have

$$\lim_{z\to-\infty} P(\hat{\theta}|z, x_I^*, 1) = 1 > c,$$

for any finite $\hat{\theta}$ and any $x_I^* \ne \infty$. We know from part (a) of Proposition 2 that the limit of $\theta^*(z)$ is finite. Therefore the indifference condition (8) cannot hold unless

$\lim_{z \to -\infty} x_I^*(z) = \infty.$ ∎

Lemma 4. For any z, there is a unique $\hat{\theta}^*$ such that, for a given x,

$$P(\hat{\theta}^*|z, x, 1) = P(\hat{\theta}^*|z, x, 0).$$

Moreover, the value of such $\hat{\theta}^*$ monotonically decreases in x.

Proof. A pair $(\hat{\theta}, x)$ solves $P(\hat{\theta}|z, x, 1) = P(\hat{\theta}|z, x, 0)$ if and only if it solves $P(\hat{\theta}|z, x, 1) = P(\hat{\theta}|z, x)$. Therefore, we define

$$G(\hat{\theta}, x) \equiv P(\hat{\theta}|z, x, 1) - P(\hat{\theta}|z, x) = \int_{-\infty}^{\hat{\theta}} \left(\frac{J(t, z)}{\int J(t, z) p(t|z, x) \, dt} - 1 \right) p(t|z, x) \, dt.$$

Since $J(t, z)$ is strictly unimodal, there exist $\theta_1(x)$ and $\theta_2(x)$ such that

$$\frac{J(t, z)}{\int J(t, z) p(t|z, x) \, dt} - 1 \begin{cases} < 0 & \text{if } t < \theta_1(x) \text{ or } t > \theta_2(x); \\ > 0 & \text{if } \theta_1(x) < t < \theta_2(x). \end{cases}$$

Thus, $G(\hat{\theta}, x)$ is negative and decreasing from 0 if $\hat{\theta} < \theta_1(x)$; increasing from $\theta_1 < \hat{\theta} < \theta_2$; and positive and decreasing towards 0 if $\hat{\theta} > \theta_2(x)$. Since $G(\hat{\theta}, x)$ is continuous, there exists a unique $\hat{\theta}^*(x) \in (\theta_1(x), \theta_2(x))$ such that $G(\hat{\theta}^*(x), x) = 0$.

To show that such $\hat{\theta}^*(x)$ is monotone decreasing in x, let $x_L < x_H$ and suppose $z \leq x_L$. We have:

$$
\begin{aligned}
G(\hat{\theta}^*(x_L), x_H) &= \int_{-\infty}^{\hat{\theta}^*(x_L)} \left(\frac{J(t, z)}{\int J(t, z) p(t|z, x_H) \, dt} - 1 \right) p(t|z, x_H) \, dt \\
&> \int_{-\infty}^{\hat{\theta}^*(x_L)} \left(\frac{J(t, z)}{\int J(t, z) p(t|z, x_L) \, dt} - 1 \right) p(t|z, x_H) \, dt \\
&> \int_{-\infty}^{\theta_1(x_L)} \left(\frac{J(t, z)}{\int J(t, z) p(t|z, x_L) \, dt} - 1 \right) p(t|z, x_L) \frac{p(\theta_1(x_L)|z, x_H)}{p(\theta_1(x_L)|z, x_L)} \, dt \\
&\quad + \int_{\theta_1(x_L)}^{\hat{\theta}^*(x_L)} \left(\frac{J(t, z)}{\int J(t, z) p(t|z, x_L) \, dt} - 1 \right) p(t|z, x_L) \frac{p(\theta_1(x_L)|z, x_H)}{p(\theta_1(x_L)|z, x_L)} \, dt \\
&= 0.
\end{aligned}
$$

The first inequality follows because $\int_{-\infty}^{\infty} J(t, z) p(t|z, x) \, dt$ is unimodal in x and attains the maximum at $x = z$. This fact is established in Lemma 2 (see Claim 1). The second inequality holds, because $p(\theta|z, x_H)/p(\theta|z, x_L)$ is increasing in θ for any given $x_H >$

x_L. To see this, we expand the posterior belief,

$$p(\theta|z, x) = \frac{p(\theta|x)p(z|x, \theta)}{p(z|x)}.$$

Therefore, for any given $x_H > x_L$, we obtain:

$$\frac{p(\theta|z, x_H)}{p(\theta|z, x_L)} = \frac{p(\theta|x_H)}{p(\theta|x_L)} \frac{p(z|x_L)}{p(z|x_H)}.$$

It is obvious that $p(\theta|x_H)/p(\theta|x_L)$ is increasing in θ.

The fact that $G(\hat{\theta}^*(x_L), x_H) > 0$ and the single-crossing property of $G(\cdot, x_H)$ implies that $\hat{\theta}^*(x_H) < \hat{\theta}^*(x_L)$. In other words, if $z \leq x_L$, $\hat{\theta}^*$ monotonically decreases in x. Similarly, for $x_L \leq z$, we show that $-G(\hat{\theta}^*(x_H), x_L) > 0$, which again implies that $\hat{\theta}^*(x_H) < \hat{\theta}^*(x_L)$. ∎

Lemma 5. For any z, there exists a unique pair $(\hat{\theta}, x)$ such that

$$P(\hat{\theta}|z, x, 1) = P(\hat{\theta}|z, x, 0) = c.$$

Proof. It is equivalent to show that, for any z, there exists a unique pair $(\hat{\theta}, x)$ such that $P(\hat{\theta}|z, x, 1) = c$ and $G(\hat{\theta}, x; z) = 0$, where G is defined in the proof of Lemma 4 above. $P(\hat{\theta}|z, x, 1)$ monotonically decreases in x and increases in $\hat{\theta}$. Therefore, for any x, there exists a unique $\hat{\theta}$ such that $P(\hat{\theta}|z, x, 1) = c$ and such value of $\hat{\theta}$ monotonically increases in x. According to Lemma 4, for any given x, there exists a unique $\hat{\theta}$ such that $G(\hat{\theta}, x; z) = 0$ and such value of $\hat{\theta}$ decreases in x. Thus the value of $\hat{\theta}$ (and hence x) that satisfies both conditions must be unique. ∎

Lemma 6. At $z = z'$ and $\hat{\theta} = \theta'$,

$$\frac{\partial \hat{x}_I}{\partial z} + \frac{\partial \hat{x}_I}{\partial \hat{\theta}} = \frac{\partial \hat{x}_m}{\partial z} + \frac{\partial \hat{x}_m}{\partial \hat{\theta}} = \frac{\partial \hat{x}_U}{\partial z} + \frac{\partial \hat{x}_U}{\partial \hat{\theta}} = 1$$

Proof. Write the relevant indifference conditions in the following form:

$$\tau_m(\hat{\theta}, z, x) \equiv P(\hat{\theta}|z, x) - c = 0,$$

$$\tau_I(\hat{\theta}, z, x) \equiv \int_{-\infty}^{\hat{\theta}} J(t, z)p(t|z, x)\,\mathrm{d}t - c\Pr[y_i = 1|z, x] = 0,$$

$$\tau_U(\hat{\theta}, z, x) \equiv \int_{-\infty}^{\hat{\theta}} (1 - J(t, z))p(t|z, x)\,\mathrm{d}t - c(1 - \Pr[y_i = 1|z, x]) = 0.$$

Claim 1. At $z = z'$ and $\hat{\theta} = \theta'$, $\partial \hat{x}_m/\partial z + \partial \hat{x}_m/\partial \hat{\theta} = 1$.

In the "mute model," we have

$$\frac{\partial \tau_m(\theta', z', x')}{\partial z} = -w\frac{1 - \beta}{\sqrt{\beta}\sigma_x}\phi\left(\frac{\theta' - X'}{\sqrt{\beta}\sigma_x}\right) + \frac{\partial w}{\partial z}\left(\Phi\left(\frac{\theta' - X'}{\sqrt{\beta}\sigma_x}\right) - \Phi\left(\frac{\theta' - x'}{\sigma_x}\right)\right)$$

$$= -w\frac{1 - \beta}{\sqrt{\beta}\sigma_x}\phi\left(\frac{\theta' - X'}{\sqrt{\beta}\sigma_x}\right),$$

where the last equality follows because $X' = \beta x' + (1 - \beta)z' = x'$. Similarly,

$$\frac{\partial \tau_m(\theta', z', x')}{\partial x} = -w\frac{\beta}{\sqrt{\beta}\sigma_x}\phi\left(\frac{\theta' - X'}{\sqrt{\beta}\sigma_x}\right) - (1 - w)\frac{1}{\sigma_x}\phi\left(\frac{\theta' - x'}{\sigma_x}\right)$$

It is straightforward to see that at the point (θ', z', x'),

$$\frac{\partial \tau_m}{\partial z} + \frac{\partial \tau_m}{\partial x} = -\frac{\partial \tau_m}{\partial \hat{\theta}}.$$

The claim then follows by the implicit function theorem.

Claim 2. At $z = z'$ and $\hat{\theta} = \theta'$,

$$\frac{\partial \tau_I}{\partial x} = J(\theta', z')\frac{\partial P(\theta'|z', x')}{\partial x} - D,$$

$$\frac{\partial \tau_U}{\partial x} = (1 - J(\theta', z'))\frac{\partial P(\theta'|z', x')}{\partial x} + D;$$

with $D < 0$.

First, note that $\partial \Pr[y_i = 1|z', x']/\partial x = 0$ by Claim 1 in the proof of Lemma 2. Rewrite τ_I using integration-by-parts and then take derivative with respect to x to get

$$D = \int_{-\infty}^{\theta'} \frac{\partial J(t, z')}{\partial t}\frac{\partial P(t|z', x')}{\partial x}\,\mathrm{d}t.$$

This term is negative because $\partial J/\partial t > 0$ for $t < \theta'$ and $\partial P/\partial x < 0$. The derivation of

$\partial \tau_U / \partial x$ follows the same lines.

Claim 3. At $z = z'$ and $\hat{\theta} = \theta'$,

$$\frac{\partial \tau_I}{\partial z} = J(\theta', z') \frac{\partial P(\theta' | z', x')}{\partial z} + Q,$$

$$\frac{\partial \tau_U}{\partial z} = (1 - J(\theta', z')) \frac{\partial P(\theta' | z', x')}{\partial z} - Q.$$

where $Q = D$.

Let $T_1 = \Pr[y_i = 1, \theta \leq \theta' | z', x', z \sim I]$, $T_2 = \Pr[y_i = 1 | z', x', z \sim I]$, $T_3 = \Pr[y_i = 1, \theta \leq \theta' | z', x', z \sim U]$, and $T_4 = \Pr[y_i = 1 | z', x', z \sim U]$. We can write

$$\tau_I(\hat{\theta}, z, x) = w(T_1 - cT_2) + (1 - w)(T_3 - cT_4).$$

Therefore,

$$\frac{\partial \tau_I(\theta', z', x')}{\partial z} = w \frac{\partial}{\partial z}(T_1 - cT_2) + (1 - w)\frac{\partial}{\partial z}(T_3 - cT_4),$$

with a term involving $\partial w / \partial z$ that vanishes because $T_1 - cT_2 = T_3 - cT_4 = 0$ when $c = 0.5$. Consider first the derivative of the term $T_1 - cT_2$:

$$\frac{\partial}{\partial z}(T_1 - cT_2) = \int_{-\infty}^{\theta'} J(t, z') \frac{1}{\sqrt{\beta}\sigma_x} \frac{\partial \phi\left(\frac{t - X'}{\sqrt{\beta}\sigma_x}\right)}{\partial z} \, dt + \int_{-\infty}^{\theta'} \frac{\partial J(t, z')}{\partial z} \frac{1}{\sqrt{\beta}\sigma_x} \phi\left(\frac{t - X'}{\sqrt{\beta}\sigma_x}\right) \, dt$$

$$- c \left[\frac{\frac{d\kappa}{dz} + \beta}{\sqrt{1 + \beta}\sigma_x} \phi\left(\frac{\overline{x}(z') - X'}{\sqrt{1 + \beta}\sigma_x}\right) - \frac{-\frac{d\kappa}{dz} + \beta}{\sqrt{1 + \beta}\sigma_x} \phi\left(\frac{\underline{x}(z') - X'}{\sqrt{1 + \beta}\sigma_x}\right) \right].$$

Use integration-by-parts on the first term to get

$$\frac{\partial}{\partial z}(T_1 - cT_2) = \frac{1}{w} J(\theta', z') \frac{\partial P(\theta' | z', x')}{\partial z}$$

$$+ \int_{-\infty}^{\theta'} \left((1 - \beta)\frac{\partial J(t, z')}{\partial t} + \frac{\partial J(t, z')}{\partial z} \right) \frac{1}{\sqrt{\beta}\sigma_x} \phi\left(\frac{t - X'}{\sqrt{\beta}\sigma_x}\right) \, dt$$

$$- c \left[\frac{\frac{d\kappa}{dz} + \beta}{\sqrt{1 + \beta}\sigma_x} \phi\left(\frac{\overline{x}(z) - X'}{\sqrt{1 + \beta}\sigma_x}\right) - \frac{-\frac{d\kappa}{dz} + \beta}{\sqrt{1 + \beta}\sigma_x} \phi\left(\frac{\underline{x}(z) - X'}{\sqrt{1 + \beta}\sigma_x}\right) \right].$$

From this, we obtain:

$$\frac{\partial}{\partial z}(T_1 - cT_2) = \frac{1}{w}J(\theta', z')\frac{\partial P(\theta'|z', x')}{\partial z}$$

$$+ \frac{\frac{d\kappa}{dz} + \beta}{\sqrt{1+\beta}\sigma_x}\phi\left(\frac{\overline{x}(z') - X'}{\sqrt{1+\beta}\sigma_x}\right)\Phi\left(\frac{\theta' - \frac{X'+\beta\overline{x}(z')}{1+\beta}}{\sqrt{\frac{\beta}{1+\beta}}\sigma_x}\right)$$

$$- \frac{-\frac{d\kappa}{dz} + \beta}{\sqrt{1+\beta}\sigma_x}\phi\left(\frac{\underline{x}(z') - X'}{\sqrt{1+\beta}\sigma_x}\right)\Phi\left(\frac{\theta' - \frac{X'+\beta\underline{x}(z')}{1+\beta}}{\sqrt{\frac{\beta}{1+\beta}}\sigma_x}\right)$$

$$- c\left[\frac{\frac{d\kappa}{dz} + \beta}{\sqrt{1+\beta}\sigma_x}\phi\left(\frac{\overline{x}(z') - X'}{\sqrt{1+\beta}\sigma_x}\right) - \frac{-\frac{d\kappa}{dz} + \beta}{\sqrt{1+\beta}\sigma_x}\phi\left(\frac{\underline{x}(z') - X'}{\sqrt{1+\beta}\sigma_x}\right)\right]$$

$$= \frac{1}{w}J(\theta', z')\frac{\partial P(\theta'|z', x')}{\partial z}$$

$$+ \frac{\beta}{\sqrt{1+\beta}\sigma_x}\phi\left(\frac{\overline{x}(z') - X'}{\sqrt{1+\beta}\sigma_x}\right)\left[\Phi\left(\frac{\theta' - \frac{X'+\beta\overline{x}(z')}{1+\beta}}{\sqrt{\frac{\beta}{1+\beta}}\sigma_x}\right) - \Phi\left(\frac{\theta' - \frac{X'+\beta\underline{x}(z')}{1+\beta}}{\sqrt{\frac{\beta}{1+\beta}}\sigma_x}\right)\right],$$

where the second equality uses the fact that $z' = \theta' = X'$ and $c = 0.5$. Similarly,

$$\frac{\partial}{\partial z}(T_3 - cT_4) = \frac{1}{\sqrt{2}\sigma_x}\phi\left(\frac{\overline{x}(z') - x'}{\sqrt{2}\sigma_x}\right)\left[\Phi\left(\frac{\theta' - \frac{x'+\overline{x}(z')}{2}}{\sqrt{\frac{1}{2}}\sigma_x}\right) - \Phi\left(\frac{\theta' - \frac{x'+\underline{x}(z')}{2}}{\sqrt{\frac{1}{2}}\sigma_x}\right)\right].$$

Combining the two terms, and using the definition of Q in the statement of the claim, we obtain:

$$Q = \frac{w\beta}{\sqrt{1+\beta}\sigma_x}\phi\left(\frac{\overline{x}(z') - X'}{\sqrt{1+\beta}\sigma_x}\right)\left[\Phi\left(\frac{\theta' - \frac{X'+\beta\overline{x}(z')}{1+\beta}}{\sqrt{\frac{\beta}{1+\beta}}\sigma_x}\right) - \Phi\left(\frac{\theta' - \frac{X'+\beta\underline{x}(z')}{1+\beta}}{\sqrt{\frac{\beta}{1+\beta}}\sigma_x}\right)\right]$$

$$+ \frac{(1-w)}{\sqrt{2}\sigma_x}\phi\left(\frac{\overline{x}(z') - x'}{\sqrt{2}\sigma_x}\right)\left[\Phi\left(\frac{\theta' - \frac{x'+\overline{x}(z')}{2}}{\sqrt{\frac{1}{2}}\sigma_x}\right) - \Phi\left(\frac{\theta' - \frac{x'+\underline{x}(z')}{2}}{\sqrt{\frac{1}{2}}\sigma_x}\right)\right] = D.$$

Thus, $\partial\tau_I/\partial z = J\partial P/\partial z + D$. Moreover, since $\tau_U = \tau_m - \tau_I$, this implies $\partial\tau_U/\partial z = (1-J)\partial P/\partial z - D$.

Claims 2 and 3 imply that

$$\frac{\partial\tau_I}{\partial x} + \frac{\partial\tau_I}{\partial z} = J\left(\frac{\partial P(\theta'|z', x')}{\partial x} + \frac{\partial P(\theta'|z', x')}{\partial z}\right).$$

From Claim 1, the term in parenthesis is equal to $-p(\theta'|z', x')$. Therefore, $\partial\tau_I/\partial x + \partial\tau_I/\partial z = -\partial\tau_I/\partial\hat{\theta}$. By the implicit function theorem, the lemma follows. ∎

Lemma 7. When z goes to ∞ or $-\infty$, $\partial \hat{x}_m / \partial \hat{\theta} < 1$.

Proof. Since $\hat{x}_m(\hat{\theta}, z)$ solves the indifference condition $P(\hat{\theta}|\hat{x}_m, z) = c$, we have $\partial \hat{x}_m / \partial \hat{\theta} = -p/P_x < 1$ if and only if $p + P_x < 0$. Expand this expression to obtain:

$$
\begin{aligned}
p + P_x &= \frac{w}{\sqrt{\beta}\sigma_x}\phi\left(\frac{\hat{\theta} - \hat{X}_m}{\sqrt{\beta}\sigma_x}\right) + \frac{1-w}{\sigma_x}\phi\left(\frac{\hat{\theta} - \hat{X}_m}{\sigma_x}\right) - \frac{-w\beta}{\sqrt{\beta}\sigma_x}\phi\left(\frac{\hat{\theta} - \hat{X}_m}{\sqrt{\beta}\sigma_x}\right) \\
&\quad - \frac{1-w}{\sigma_x}\phi\left(\frac{\hat{\theta} - \hat{X}_m}{\sigma_x}\right) + \frac{\partial w}{\partial x}\left[\Phi\left(\frac{\hat{\theta} - \hat{X}_m}{\sqrt{\beta}\sigma_x}\right) - \Phi\left(\frac{\hat{\theta} - \hat{X}_m}{\sigma_x}\right)\right] \\
&= \frac{w(1-\beta)}{\sqrt{\beta}\sigma_x}\phi\left(\frac{\hat{\theta} - \hat{X}_m}{\sqrt{\beta}\sigma_x}\right) + \frac{\partial w}{\partial x}\left[\Phi\left(\frac{\hat{\theta} - \hat{X}_m}{\sqrt{\beta}\sigma_x}\right) - \Phi\left(\frac{\hat{\theta} - \hat{X}_m}{\sigma_x}\right)\right] \\
&= w\left\{\frac{1-\beta}{\sqrt{\beta}\sigma_x}\phi\left(\frac{\hat{\theta} - \hat{X}_m}{\sqrt{\beta}\sigma_x}\right) + (1-w)\left(\frac{z - \hat{X}_m}{\sigma_x}\right)\left[\Phi\left(\frac{\hat{\theta} - \hat{X}_m}{\sqrt{\beta}\sigma_x}\right) - \Phi\left(\frac{\hat{\theta} - \hat{X}_m}{\sigma_x}\right)\right]\right\}.
\end{aligned}
$$

When $z \to +\infty$, the expression in curly brackets goes to

$$
-(1-w)\left(\frac{z - \hat{X}_m}{\sigma_x}\right)\Phi\left(\frac{\hat{\theta} - \hat{X}_m}{\sigma_x}\right) < 0.
$$

Similarly, when $z \to -\infty$, it goes to

$$
(1-w)\left(\frac{z - \hat{X}_m}{\sigma_x}\right)\left(1 - \Phi\left(\frac{\hat{\theta} - \hat{X}_m}{\sigma_x}\right)\right) < 0. \qquad \blacksquare
$$

Lemma 8. For z sufficiently large,

$$
\left(1 - J(\hat{\theta}, z)\right)\Phi\left(\frac{\hat{x}_U(\hat{\theta}, z) - \hat{\theta}}{\sigma_x}\right) > \Phi\left(\frac{\hat{x}_m(\hat{\theta}, z) - \hat{\theta}}{\sigma_x}\right).
$$

For z sufficiently negative, this inequality is reversed.

Proof. We establish this lemma in several steps.

Claim 1. For z sufficiently large,

$$
\lim_{z \to \infty} \frac{P\left(\hat{\theta}|z, x, 0\right) - P\left(\hat{\theta}|z, x\right)}{J\left(\hat{\theta}, z\right)} = \infty.
$$

Denote $\bar{J} = \Pr[y = 1|z, x]$. We have:

$$\bar{J} = w \left(\Phi \left(\frac{\overline{x}(z) - X}{\sqrt{1 + \beta}\sigma_x} \right) - \Phi \left(\frac{\underline{x}(z) - X}{\sqrt{1 + \beta}\sigma_x} \right) \right) + (1 - w) \left(\Phi \left(\frac{\overline{x}(z) - x}{\sqrt{2}\sigma_x} \right) - \Phi \left(\frac{\underline{x}(z) - x}{\sqrt{2}\sigma_x} \right) \right).$$

Therefore,

$$
\begin{aligned}
\frac{P(\hat{\theta}|z, x, 0) - P(\hat{\theta}|z, x)}{J(\hat{\theta}, z)} &= \frac{\int_{-\infty}^{\hat{\theta}} (1 - J(t, z)) \, p(t|z, x) \, \mathrm{d}t - (1 - \bar{J})P(\hat{\theta}|z, x)}{(1 - \bar{J})J(\hat{\theta}, z)} \\
&> \frac{\bar{J}P(\hat{\theta}|z, x) - J(\hat{\theta}, z)P(\hat{\theta}|z, x)}{(1 - \bar{J})J(\hat{\theta}, z)} \\
&= \frac{P(\hat{\theta}|z, x)}{1 - \bar{J}} \left(\frac{\bar{J}}{J} - 1 \right).
\end{aligned}
$$

The inequality follows, because $J(t, z) < J(\hat{\theta}, z)$ for any $t < \hat{\theta} < z$. The limit of $P(\hat{\theta}|z, x)/(1 - \bar{J})$ is $\Phi(\sigma_x^{-1}(\hat{\theta} - x))$, which is bounded above zero. Moreover,

$$\lim_{z \to \infty} \frac{\bar{J}}{J} = \lim_{z \to \infty} w \frac{1 - \Phi\left(\frac{x(z) - X}{\sqrt{1 + \beta}\sigma_x} \right)}{1 - \Phi\left(\frac{x(z) - \hat{\theta}}{\sigma_x} \right)} + (1 - w) \frac{1 - \Phi\left(\frac{x(z) - x}{\sqrt{2}\sigma_x} \right)}{1 - \Phi\left(\frac{x(z) - \hat{\theta}}{\sigma_x} \right)}.$$

The second term is unbounded since the numerator goes to 0 at the rate slower than does the denominator. This establishes the claim.

Claim 2. For z sufficiently large,

$$\lim_{z \to \infty} \frac{\hat{x}_U(\hat{\theta}, z) - \hat{x}_m(\hat{\theta}, z)}{J(\hat{\theta}, z)} = \infty.$$

For any $\hat{\theta}$, $P(\hat{\theta}|z, \hat{x}_U, 0) = c$ and $P(\hat{\theta}|z, \hat{x}_m) = c$. Using a first-order expansion, we obtain:

$$\hat{x}_U - \hat{x}_m = \frac{P(\hat{\theta}|\hat{x}_U, z) - P(\hat{\theta}|\hat{x}_U, z, 0)}{P_x(\hat{\theta}|z, x)},$$

for some $x \in (\hat{x}_m, \hat{x}_U)$. Thus,

$$\frac{\hat{x}_U(\hat{\theta}, z) - \hat{x}_m(\hat{\theta}, z)}{J(\hat{\theta}, z)} = \frac{-1}{P_x(\hat{\theta}|z, x)} \left(\frac{P(\hat{\theta}|\hat{x}_U, z, 0) - P(\hat{\theta}|\hat{x}_U, z)}{J(\hat{\theta}, z)} \right).$$

It is easy to verify that $P_x(\hat{\theta}|z, x)$ is bounded away from 0, because

$$\lim_{z \to \infty} P_x(\hat{\theta}|z, x) = \frac{-1}{\sigma_x} \phi \left(\frac{\hat{\theta} - x}{\sigma_x} \right) < 0.$$

The claim then follows from Claim 1.

The inequality in the lemma holds if and only if

$$\Phi\left(\frac{\hat{x}_U - \hat{\theta}}{\sigma_x}\right) - \Phi\left(\frac{\hat{x}_m - \hat{\theta}}{\sigma_x}\right) > J\Phi\left(\frac{\hat{x}_U - \hat{\theta}}{\sigma_x}\right).$$

Using a first-order expansion, this is equivalent to

$$\phi\left(\frac{x - \hat{\theta}}{\sigma_x}\right)\left(\frac{\hat{x}_U - \hat{x}_m}{\sigma_x}\right) > J\Phi\left(\frac{\hat{x}_U - \hat{\theta}}{\sigma_x}\right),$$

for some $x \in (\hat{x}_m, \hat{x}_U)$. Rearrange it, we observe that it is equivalent to

$$\frac{\frac{1}{\sigma_x}\phi\left(\frac{x - \hat{\theta}}{\sigma_x}\right)}{\Phi\left(\frac{\hat{x}_U - \hat{\theta}}{\sigma_x}\right)} > \frac{J}{\hat{x}_U - \hat{x}_m}.$$

The left-hand-side is positive and bounded above 0. Claim 2 establishes that the right-hand-side goes to 0, as z goes to infinity. For the case of z going to negative infinity, the argument is symmetric. ∎

Lemma 9. Let $L \equiv D^2/[(JP_x - D)((1 - J)P_x + D)]$. There exist values of δ for which the lower bound of L is increasing in β and is bounded above 0 when β approaches 1.

Proof. We take a number of steps to prove this lemma.

Claim 1. $D < fP_x$, where

$$f \equiv \sqrt{\frac{\beta}{1 + \beta}} \frac{\phi\left(\frac{\kappa}{\sqrt{1+\beta}\sigma_x}\right)}{\phi(0)}\left[2\Phi\left(\frac{\kappa}{\sqrt{\frac{1+\beta}{\beta}}\sigma_x}\right) - 1\right] > 0.$$

Using the formula for D in the proof of Lemma 6,

$$-D = \frac{w\beta}{\sqrt{1+\beta}\sigma_x}\phi\left(\frac{\kappa}{\sqrt{1+\beta}\sigma_x}\right)\left[2\Phi\left(\frac{\kappa}{\sqrt{\frac{1+\beta}{\beta}}\sigma_x}\right) - 1\right]$$

$$+ \frac{1-w}{\sqrt{2}\sigma_x}\phi\left(\frac{\kappa}{\sqrt{2}\sigma_x}\right)\left[2\Phi\left(\frac{\kappa}{\sqrt{2}\sigma_x}\right) - 1\right]$$

$$= f\left[\frac{w\beta}{\sqrt{\beta}\sigma_x}\phi(0) + \frac{1-w}{\sigma_x}\phi(0)\sqrt{\frac{1+\beta}{2\beta}}\frac{\phi\left(\frac{\kappa}{\sqrt{2}\sigma_x}\right)}{\phi\left(\frac{\kappa}{\sqrt{1+\beta}\sigma_x}\right)}\frac{2\Phi\left(\frac{\kappa}{\sqrt{2}\sigma_x}\right)-1}{2\Phi\left(\frac{\kappa}{\sqrt{\frac{1+\beta}{\beta}}\sigma_x}\right)-1}\right]$$

$$> f\left[\frac{w\beta}{\sqrt{\beta}\sigma_x}\phi(0) + \frac{1-w}{\sigma_x}\phi(0)\right]$$

$$= -fP_x.$$

The inequality holds because $\beta < 1$ and therefore,

$$\sqrt{\frac{1+\beta}{2\beta}}\frac{\phi\left(\frac{\kappa}{\sqrt{2}\sigma_x}\right)}{\phi\left(\frac{\kappa}{\sqrt{1+\beta}\sigma_x}\right)}\frac{2\Phi\left(\frac{\kappa}{\sqrt{2}\sigma_x}\right)-1}{2\Phi\left(\frac{\kappa}{\sqrt{\frac{1+\beta}{\beta}}\sigma_x}\right)-1} > 1.$$

This shows that $D < fP_x$.

Claim 2. Let $T(f, J) \equiv f^2/[(J-f)(1-J+f)]$. Then, $L > T(f, J) > 0$.

Since $\partial L/\partial D < 0$ and $D < fP_x$ (Claim 1), we obtain:

$$\frac{D^2}{(JP_x - D)((1-J)P_x + D)} > \frac{(fP_x)^2}{(JP_x - fP_x)((1-J)P_x + fP_x)}$$

$$= \frac{f^2}{(J-f)((1-J)+f)}$$

$$> 0.$$

The last inequality holds because $JP_x - D < 0$ implies that $J > f$.

We now show that there always exists δ such that $T(f, J)$ is increasing in β and is bounded above 0 when β approaches 1.

Take $\kappa = k_0 \sigma_x$ for some constant k_0. Then

$$f = \sqrt{\frac{\beta}{1+\beta}} \frac{\phi\left(\frac{k_0}{\sqrt{1+\beta}}\right)}{\phi(0)} \left[2\Phi\left(\frac{k_0}{\sqrt{\frac{1+\beta}{\beta}}}\right) - 1\right],$$

$$J = 2\Phi(k_0) - 1.$$

For such choice of κ, $T(f, J)$ depends on β only through f. It can be verified that f is increasing in β and T is increasing in f. Hence the lower bound T is increasing in β. Moreover, for such choice of κ, T is bounded above 0 when β equals 1. Finally, κ is decreasing in δ, with κ approaches infinity when δ approaches zero and with κ equal to zero when δ is sufficiently high. Therefore, there exists a corresponding value of δ for any choice of κ. ∎

Lemma 10. In the "communication model," the equilibrium triple (θ^*, x_I^*, x_U^*) has the following properties: $\lim_{z \to \pm\infty} \theta^*(z) = \theta_{ms}^*$, $\lim_{z \to \pm\infty} x_U^*(z) = x_{ms}^*$, and $\lim_{z \to \pm\infty} x_I^*(z) = \mp\infty$.

Proof. We only prove the limiting properties of $x_U^*(z)$ and $\theta^*(z)$ here. The limiting properties of $x_I^*(z)$ require comparing the rates of convergence of different functions, and they are formally established in Lemma 3.

Suppose the limit values of both $x_U^*(z)$ and $\theta^*(z)$ are finite. The indifference condition (9) in "communication model" requires

$$\lim_{z \to \infty} \int_{-\infty}^{\theta^*} \frac{1 - J(t, z)}{\Pr[y_i = 0 | z, x_U^*]} p(t | z, x_U^*) \, dt = c.$$

By Lemma 3 (Claim 1), both $\underline{x}(z)$ and $\overline{x}(z)$ go to infinity as z goes to infinity. Therefore, for any $t \le \theta^*$, the probability that x_j does not belong to $[\underline{x}(z), \overline{x}(z)]$ goes to one. We thus have

$$\lim_{z \to \infty} \frac{1 - J(t, z)}{\Pr[y_i = 0 | z, x_U^*]} = 1.$$

The indifference condition for type x_U^* becomes:

$$\lim_{z \to \infty} \int_{-\infty}^{\theta^*} p(t | z, x_U^*) \, dt = \Phi\left(\frac{\theta^* - x_U^*}{\sigma_x}\right) = c,$$

where the first equality follows because $w(z, x_U^*)$ goes to 0 as z goes to infinity.

When z goes to infinity, $J(\theta^*, z)$ goes to zero. Therefore the critical mass condition

(10) for the "communication model" becomes,

$$\lim_{z \to \infty} J(\theta^*, z) \Phi \left(\frac{x_I^* - \theta^*}{\sigma_x} \right) + (1 - J(\theta^*, z)) \Phi \left(\frac{x_U^* - \theta^*}{\sigma_x} \right) = \Phi \left(\frac{x_U^* - \theta^*}{\sigma_x} \right) = \theta^*.$$

Given (θ^*, x_U^*) solves the same equation system as that in the "pure noise model," we conclude that $\lim_{z \to \infty} x_U^*(z) = x_{ms}^*$ and $\lim_{z \to \infty} \theta^*(z) = \theta_{ms}^*$. The proof of the case for the limit as z goes to minus infinity is analogous. ∎

B. The Participation Cost in the "Communication Model"

Our analysis of the "communication model" focuses on two types of rumors: (1) the case when the rumor is extreme (Proposition 3); and (2) the case when the rumor is close to neutral (Proposition 4 and 5). Proposition 3 holds regardless of the cost of participating in attack c. Proposition 4 and 5 are proved using the parameter restriction $c = 0.5$. This case facilitates comparison across models and offers analytical convenience for the following reasons.

First, when $c = 0.5$, we have $x' = \theta' = z' = 0.5$ for any parameter values of the model. This property implies that the conditional mean of the regime strength θ is the same across models:

$$E[\theta | x'] = E[\theta | z', x'] = E[\theta | z', x', 1] = E[\theta | z', x', 0] = \theta'.$$

This common posterior mean allows us to use the ordering of derivatives with respect to z to make inference on the ranking of equilibrium regime thresholds across models, and therefore, to analyze the effects of public signal, skepticism and communication.

Second, in this case, we shut down one aspect of the "skepticism effect" when z is close to the neutral rumor, which simplifies our analysis substantially. When the value of z deviates from z', the "skepticism effect" manifests itself in two facts—that citizens are not sure if the rumor is informative or not ($w(z, x_i)$ is between 0 and 1), and that citizens adjust the posterior weight when z adjusts ($\partial w(z, x_i)/\partial z$ may or may not be equal to 0). The second effect is very small when we vary z around the neutral rumor. The intuition is that when the rumor is neutral, it does not matter that much to citizens whether it is from the informative or uninformative source in the "mute model." It is not that important either, whether the messages from their peers confirm the rumor or not in "communication model." Therefore, the effect arising from the change in $w(z, x_i)$ tends to be small. In the case where $c = 0.5$, we obtain $x' = z'$, therefore, the second channel of skepticism is closed down, i.e., $\partial w(z', x')/\partial z = 0$. But as long as x' is not very far away from z', $\partial w(z', x')/\partial z$ will be small.

In what follows, we demonstrate that removing such an effect does not affect our results when rumor is close to neutral. Proposition 4 is proved analytically for the case $c = 0.5$. But we verify numerically that the result holds generally for any c.

Conjecture. For any c, at $z = z'$ such that $\theta^*(z') = \theta_m^*(z')$,

$$\frac{\mathrm{d}\theta^*(z')}{\mathrm{d}z} < \frac{\mathrm{d}\theta_m^*(z')}{\mathrm{d}z} < 0.$$

Following the same logic described in the text, the above conjecture is true if conditions (17) and (18) hold for any c. We can show analytically that (18) holds for any c. We provide a sufficient condition for (17) to hold, and verify numerically that such a sufficient condition is satisfied for any c.

Lemma 11. For any c, at $z = z'$ and $\hat{\theta} = \theta'$,

$$J\frac{\partial \hat{x}_I}{\partial \hat{\theta}} + (1 - J)\frac{\partial \hat{x}_U}{\partial \hat{\theta}} > \frac{\partial \hat{x}_m}{\partial \hat{\theta}}.$$

Proof. For any c, we derive $\partial \hat{x}_I / \partial \hat{\theta}$ and $\partial \hat{x}_U / \partial \hat{\theta}$ as follows,

$$\frac{\partial \hat{x}_I}{\partial \hat{\theta}} = \frac{-Jp(\theta'|z', x')}{JP_x - D} = \frac{\frac{\partial \hat{x}_m}{\partial \hat{\theta}}}{1 - \frac{D}{JP_x}},$$

$$\frac{\partial \hat{x}_U}{\partial \hat{\theta}} = \frac{-(1 - J)p(\theta'|z, x')}{(1 - J)P_x + D} = \frac{\frac{\partial \hat{x}_m}{\partial \hat{\theta}}}{1 + \frac{D}{(1-J)P_x}}.$$

where $D = \int_{-\infty}^{\theta'} P_x dJ(t, z') + c \cdot \partial \Pr[y = 1|z', x'] / \partial x$. Inequality (18) holds, because

$$J\frac{\partial \hat{x}_I}{\partial \hat{\theta}} + (1 - J)\frac{\partial \hat{x}_U}{\partial \hat{\theta}} - \frac{\partial \hat{x}_m}{\partial \hat{\theta}} = \left[\frac{J}{1 - \frac{D}{JP_x}} + \frac{1 - J}{1 + \frac{D}{(1-J)P_x}} - 1\right]\frac{\partial \hat{x}_m}{\partial \hat{\theta}}$$

$$= \left(\frac{D}{P_x}\right)^2 \frac{\frac{\partial \hat{x}_I}{\partial \hat{\theta}}\frac{\partial \hat{x}_U}{\partial \hat{\theta}}}{J}(1 - J)\frac{\partial \hat{x}_m}{\partial \hat{\theta}} > 0. \qquad \blacksquare$$

Lemma 12. For any c, at $z = z'$ and $\hat{\theta} = \theta'$,

$$\frac{\partial \hat{x}_I}{\partial \hat{\theta}} > \frac{\partial \hat{x}_m}{\partial \hat{\theta}} > \frac{\partial \hat{x}_U}{\partial \hat{\theta}}.$$

Hence, $D < 0$ for any c.

Proof. Toward a contradiction, assume that $\partial \hat{x}_I / \partial \hat{\theta} < \partial \hat{x}_m / \partial \hat{\theta}$, evaluated at $z = z'$ and $\hat{\theta} = \theta'$. Pick θ_1 slightly above θ'. There exists $x_1 > x'$ such that $P(\theta_1 | z', x_1, 1) = c$. From the proof of Lemma 4, we know that there exists θ_2 such that $G(\theta_2, x_1) = 0$ and $\theta_2 < \theta'$. Therefore, $\theta_1 > \theta_2$. Since $G(\cdot, x_1)$ crosses zero once and from above, it follows that $G(\theta_1, x_1) > 0$. This implies that

$$P(\theta_1 | z', x_1) < P(\theta_1 | z', x_1, 1) = c.$$

Let x_2 be such that $P(\theta_1 | z', x_2) = c$. By assumption, $\partial \hat{x}_I / \partial \hat{\theta} < \partial \hat{x}_m / \partial \hat{\theta}$. Therefore, $x_2 = \hat{x}_m(\theta_1, z') > \hat{x}_I(\theta_1, z') = x_1$. Since $P(\hat{\theta} | z', x)$ is decreasing in x, we obtain:

$$P(\theta_1 | z', x_1) > P(\theta_1 | z', x_2) = c,$$

a contradiction. Thus, we must have $\partial \hat{x}_I / \partial \hat{\theta} > \partial \hat{x}_m / \partial \hat{\theta}$. The proof that $\partial \hat{x}_m / \partial \hat{\theta} > \partial \hat{x}_U / \partial \hat{\theta}$ follows a similar logic. Finally,

$$\frac{\partial \hat{x}_I}{\partial \hat{\theta}} - \frac{\partial \hat{x}_m}{\partial \hat{\theta}} = \frac{-pD}{(JP_x - D)P_x}.$$

Since $JP_x - D$ and P_x are negative, $\partial \hat{x}_I / \partial \hat{\theta} > \partial \hat{x}_m / \partial \hat{\theta}$ implies that $D < 0$. $\blacksquare$

It remains to be shown that inequality (17) holds for any c. We can write:

$$\frac{\partial \hat{x}_I}{\partial z} = \frac{\frac{\partial \hat{x}_m}{\partial z}}{1 - \frac{D}{JP_x}} + \frac{-\frac{Q}{JP_x}}{1 - \frac{D}{JP_x}}, \qquad \frac{\partial \hat{x}_U}{\partial z} = \frac{\frac{\partial \hat{x}_m}{\partial z}}{1 + \frac{D}{(1-J)P_x}} + \frac{\frac{Q}{(1-J)P_x}}{1 + \frac{D}{(1-J)P_x}};$$

where $Q = \partial \tau_I / \partial z - J(\theta', z') \partial P(\theta' | z', x') / \partial z$. Since $\partial \hat{x}_m(\theta', z') / \partial z < 0$, following the same logic as in the proof of Lemma 11, we have

$$J \left(\frac{\frac{\partial \hat{x}_m}{\partial z}}{1 - \frac{D}{JP_x}} \right) + (1 - J) \left(\frac{\frac{\partial \hat{x}_m}{\partial z}}{1 + \frac{D}{(1-J)P_x}} \right) < \frac{\partial \hat{x}_m}{\partial z}.$$

Therefore, inequality (17) holds if

$$\frac{pDQ}{(-P_x)(JP_x - D)((1-J)P_x + D)} > 0.$$

Lemma 12 already establishes that $D < 0$. Therefore, it suffices to show that $Q < 0$, which is sufficient for the conjecture to hold. We have shown that $Q = D < 0$, when $c = 0.5$. When $c \neq 0.5$, an analytical proof cannot be obtained. Numerically, we verify that $Q < 0$ holds for a wide range of parameter values. For example, our computation shows negative values of Q on the grids (c, s), with $c \in \{0.1, 0.2, \ldots, 0.9\}$

and $s \in \{-1.5, -1.4, \ldots, 1.5\}$, using other parameter values specified in Section 3.